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Constitutive equations optimized for determining strengths of metallic alloys

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ABSTRACT

We investigate compatibilities of three constitutive equations, the Hollomon, the Swift, and the Voce equations for determination of yield and ultimate tensile strengths based on tensile true stress–strain curves of 27 metal alloys including those with power-law type and linear-type strain-hardening. We analyze each constitutive equation in terms of yield strength determined by the intercept of the linear elastic loading curve and plastic flow curve and ultimate tensile strength evaluated by the concept of instability in tension. We found that the describing plastic flow is very sensitive in determination of the yield strength and tensile strength from parameters of constitutive equation. Voce equation gives estimate yield strength and tensile strength better than Hollomon and Swift equations.

1. Introduction

Tensile testing generally produces uniaxial stress–strain relations of materials that are useful enough to predict multi-axial elasto-plastic deformation through conventional mechanics models (Dieter, 1961; von Mises, 1913). The stress–strain relation in the plastic flow region is described by a constitutive equation containing some number of parameters corresponding to tensile properties. The most commonly used constitutive equation is the Hollomon equation, which has the form of a power-law relation between true stress σ and strain ε;

$$\sigma = K\varepsilon^n, \tag{1}$$

where K is the strength coefficient and n is the strain-hardening exponent (Dieter, 1961; Hollomon, 1945; Kleemola and Nieminen, 1974). The yield strength can be determined as the intercept of elastic linear line and plastic flow curve described by the constitutive equation (Dieter, 1961; ASTM, 2002). If the plastic flow curve is well described by Eq. (1), the yield strength can be determined using the elastic modulus E, K and n. The ultimate tensile strength $(\varepsilon_u)(\sigma_u)$ at which necking initiates can be determined using the instability conditions for the Hollomon equation:

$$\frac{\partial\sigma}{\partial\varepsilon} = \sigma, \tag{2}$$

$$\varepsilon_u = n, \tag{3}$$

$$\sigma_u = K\varepsilon_u^n = Kn^n, \tag{4}$$

where ε_u is the strain at ultimate tensile stress (Dieter, 1961; Kim et al., 2006a). From the equations above, we can determine the yield strength and the ultimate tensile

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strength analytically for materials well described by the Hollomon equation if E, K, and n are known.

Though yield and ultimate tensile strengths are directly measured in uniaxial tensile testing, constitutive equations play an important role in evaluating tensile stress–strain curves from other indirect methods such as instrumented indentation testing and small punch testing (Kim et al., 2006a; Ahn and Kwon, 2001; Kim et al., 2006b; Buclille et al., 2003; Dao et al., 2001; Lee et al., 2010, 2008; Foulds et al., 1998; Fleury and Ha, 1998; Guillemot et al., 2012). In these tests, flow stress–strain points are not measured for the whole range of the tensile curve, and the ultimate tensile strength cannot be measured due to absence of necking. Thus, tensile properties, yield and ultimate tensile strengths, K and n must be determined using constitutive equations in an analytical way (Kim et al., 2006a; Ahn and Kwon, 2001; Kim et al., 2006b; Lee et al., 2008). The optimum choice and accuracy of constitutive equation is critical in determining tensile strengths in such tests. In addition to the Hollomon equation, other constitutive equations (Yoo and Park, 2008; Swift, 1952; Ludwigson, 1971; Samuel and Rodriguez, 2005; Voce, 1948, 1955; Kim et al., 2013) have been suggested to describe some materials which do not show power-law relation (i.e. austenitic stainless steels show linear-type strain hardening (Lee et al., 2008). Swift and Voce equations are also well known for constitutive equation describing plastic hardening behaviors of metal, where the Swift equation is given by

$$\sigma = K_s(\varepsilon + \varepsilon_0)^{n_s}; \quad (5)$$

the Voce equation is

$$\sigma = \sigma_0 - A\sigma_0 \exp(-\beta\varepsilon). \quad (6)$$

K_s, ε_0 n_s are constants for Swift equation and σ_0, A, β are for Voce equation. Here, we suggest the optimal among three constitutive equations, the Hollomon, the Swift and the Voce equations, for seventeen metallic alloys with power-law hardening and ten alloys with linear hardening in terms of analytical determination of tensile strengths.

2. Experiments

Cylindrical tensile samples with gauge length 25 mm and diameter 6 mm were prepared for 27 metal alloys, SKH51, SK4, SCM21, SCM4, API X100, API X120 (carbon steels), STS403, STS410, STS420J2, STS440 (ferrite-based stainless steels), STS303, STS304, STS304L, STS310S, STS316, STS316L, STS321, STS347 (austenite-based stainless steels), Inconel 600, Inconel 825 (Ni alloys), Ti–5Al–2.5Sn, Ti–6Al–4V, Ti–6Al–6V–2Sn, Ti–7Al–4Mo (Ti alloys), Al2024, Al6061, and Al7075 (Al alloys). Uniaxial tensile tests were carried out using Instron 5582 (Instron Inc., USA) at cross-head speed 1 mm/min as per the ASTM standard (ASTM, 2002). Measured true stress and strain data were analyzed using the commercial software OriginPro 7.5 SR0 (OriginLab Co., MA). Power-law type equations were used for the Hollomon and Swift equations and exponential-type equations for the Voce equation. The data fitting was performed with least square regression and data points are

Table 1
Tensile properties; elastic modulus was measured by ultrasonic, yield strength was measured by the 0.2% offset method, ultimate tensile strength was measured by maximum stress point in engineering stress–strain curve, and n and K were evaluated by the Hollomon equation.

Sample	Elastic modulus (GPa)	Yield		Ultimate tensile	
		Strain	Stress (MPa)	Strain	Stress (MPa)
SKH51 (tempered)	223	0.0033	280	0.1181	785
SK4 (tempered)	204	0.0040	409	0.1638	750
SCM21 (tempered)	194	0.0034	275	0.1476	649
SCM4 (tempered)	177	0.0060	716	0.0667	999
API X100	203	0.0049	592	0.0891	913
API X120	210	0.0057	749	0.0532	1025
STS403 (tempered)	211	0.0036	330	0.1563	674
STS410 (tempered)	215	0.0037	357	0.1521	672
STS420J2 (tempered)	211	0.0038	392	0.1305	806
STS440 (annealed)	220	0.0035	323	0.1167	817
STS303 (annealed)	206	0.0036	314	0.4822	1082
STS304 (annealed)	190	0.0036	311	0.5374	1163
STS304L (annealed)	203	0.0037	338	0.4900	1116
STS310S (annealed)	192	0.0034	258	0.3468	780
STS316 (annealed)	198	0.0034	282	0.5014	1061
STS316L (annealed)	198	0.0035	305	0.4524	949
STS321 (annealed)	197	0.0036	305	0.4512	911
STS347 (solid solution heat treatment)	200	0.0031	211	0.4169	884
Inconel 600 (annealed)	170	0.0044	405	0.2715	985
Inconel 825 (annealed)	161	0.0041	341	0.3097	984
Ti–5Al–2.5Sn (solution treatment)	130	0.0086	863	0.0903	1029
Ti–6Al–4V (solution treatment)	110	0.0104	930	0.0858	1097
Ti–6Al–6V–2Sn (solution treatment)	122	0.0102	998	0.1070	1174
Ti–7Al–4Mo (solution treatment)	132	0.0098	1031	0.0802	1134
Al2024 (tempered)	72	0.0084	461	0.1283	670
Al6061 (tempered)	70	0.0057	259	0.0508	298
Al7075 (tempered)	71	0.0094	524	0.0828	626

Table 2
Parameters in the Hollomon, Swift, and Voce equations fitted to stress–strain points in the plastic flow region, yield and ultimate tensile strengths analyzed by the constitutive equations, their errors compared to data, and coefficient of determination (R) in Table 1.

Sample	Hollomon							Swift								Voce							
	n	K	σ_y (MPa)	Error (%)	σ_u (MPa)	Error (%)	R	K_s	ε	n_s	ε_y (MPa)	Error (%)	σ_u (MPa)	Error (%)	R	σ_0	σ_0 A	β	σ_y (MPa)	Error (%)	σ_u (MPa)	Error (%)	R
SKH51	0.2586	1458	339	21	1027	31	0.9694	1249	−0.0032	0.1997	–	–	906	15	0.9880	768	520	41.93	318	14	750	−4	0.9934
SK4	0.1797	1043	383	−6	766	2	0.9986	1064	0.0016	0.1893	398	−3	777	4	0.9992	765	362	16.57	427	4	722	−4	0.9975
SCM21	0.2224	1029	294	7	737	14	0.9935	961	−0.002	0.1928	269	−2	700	8	0.9979	647	365	25.49	315	15	623	−4	0.9930
SCM4	0.1296	1435	743	4	1101	10	0.9942	1348	−0.0032	0.1075	718	0	1061	6	0.9990	1008	355	45.8	740	3	986	−1	0.9973
API X100	0.1415	1303	615	4	989	8	0.9942	1236	−0.0026	0.1191	603	2	960	5	0.9993	919	355	39.02	628	6	896	−2	0.9940
API X120	0.1299	1532	786	5	1175	15	0.9745	1332	−0.0046	0.0857	741	−1	1079	5	0.9988	1020	390	77.26	771	3	1007	−2	0.9937
STS403	0.2124	1010	303	−8	727	8	0.9942	763	−0.0039	0.1112	–	–	597	−11	0.9958	643	361	26.86	314	−5	620	−8	0.9995
STS410	0.1786	936	343	−4	688	2	0.9952	986	0.0036	0.2026	–	–	713	6	0.9983	693	330	16.13	383	7	653	−3	0.9987
STS420J2	0.2073	1261	400	2	910	13	0.9949	1219	−0.0012	0.1938	386	−1	887	10	0.9957	806	426	25.95	423	8	776	−4	0.9985
STS440	0.2565	1508	357	11	1064	30	0.9760	1303	−0.0033	0.202	–	–	943	15	0.9898	805	520	37.85	350	8	784	−4	0.9955
STS303	0.3271	1256	187	−40	871	−19	0.9718	1541	0.0829	0.6146	341	9	1143	6	0.9999	1851	1507	1.4	354	13	1080	0	0.9998
STS304	0.3482	1303	170	−45	903	−22	0.9721	1568	0.1019	0.6796	341	10	1206	4	0.9998	2499	2161	0.89	345	11	1174	1	0.9994
STS304L	0.32	1304	202	−40	905	−19	0.9791	1538	0.0618	0.5354	358	6	1101	−1	0.9998	1646	1280	1.78	374	11	1054	−5	0.9991
STS310S	0.2939	1014	183	−29	707	−9	0.9794	1226	0.0336	0.4551	274	6	857	10	0.9992	993	716	3.49	286	11	772	−1	0.9997
STS316	0.3459	1241	161	−43	859	−19	0.9805	1479	0.0634	0.5842	305	8	1080	2	0.9999	1686	1375	1.55	320	13	1024	−3	0.9995
STS316L	0.3086	1124	186	−39	782	−18	0.9774	1337	0.0613	0.526	316	4	953	0	0.9999	1389	1065	1.9	331	9	910	−4	0.9994
STS321	0.2923	1063	193	−37	742	−19	0.9664	1314	0.0789	0.5539	330	8	947	4	0.9998	1397	1065	1.81	340	11	899	−1	0.9997
STS347	0.3653	1137	130	−38	787	−11	0.9875	1356	0.0333	0.5385	228	8	972	10	0.9997	1238	1001	2.43	244	16	877	−1	0.9997
Inconel 600	0.2654	1331	303	−25	936	−5	0.9786	1655	0.0341	0.4299	407	0	1151	17	0.9995	1257	853	4.2	420	4	1016	3	0.9998
Inconel 825	0.309	1318	226	−34	917	−7	0.9758	1707	0.0457	0.5287	350	3	1219	24	0.9995	1441	1094	2.8	359	5	1062	8	0.9998
Ti–5Al–2.5Sn	0.0662	1202	882	2	1004	−2	0.9892	1311	0.0263	0.113	898	4	1025	0	0.9910	1114	237	11.85	898	4	1028	0	0.9847
Ti–6Al–4V	0.0802	1336	930	0	1092	−1	0.9984	1362	0.0034	0.0888	930	0	1099	0	0.9996	1134	247	21.95	930	0	1085	−1	0.9988
Ti–6Al–6V–2Sn	0.076	1379	970	−3	1134	−3	0.9769	1562	0.0391	0.1489	998	0	1177	0	0.9997	1311	336	8.53	998	0	1174	0	0.9996
Ti–7Al–4Mo	0.0548	1291	1003	−3	1101	−3	0.9501	640	1.262	1.9765	1031	0	1013	−11	0.9997	129	−882	−2.24	1031	0	1011	−11	0.9999
Al2024	0.1503	907	439	−5	682	2	0.9931	999	0.0124	0.2014	456	−1	723	8	0.9995	726	293	12.68	464	1	673	1	0.9999
Al6061	0.0625	361	262	1	304	2	0.9951	382	0.0058	0.0799	268	4	312	5	0.9978	314	56	35.07	269	4	306	3	0.9947
Al7075	0.0804	757	524	0	618	−1	0.9952	793	0.0079	0.1026	524	0	628	0	0.9994	649	148	19.29	524	0	617	−1	0.9985

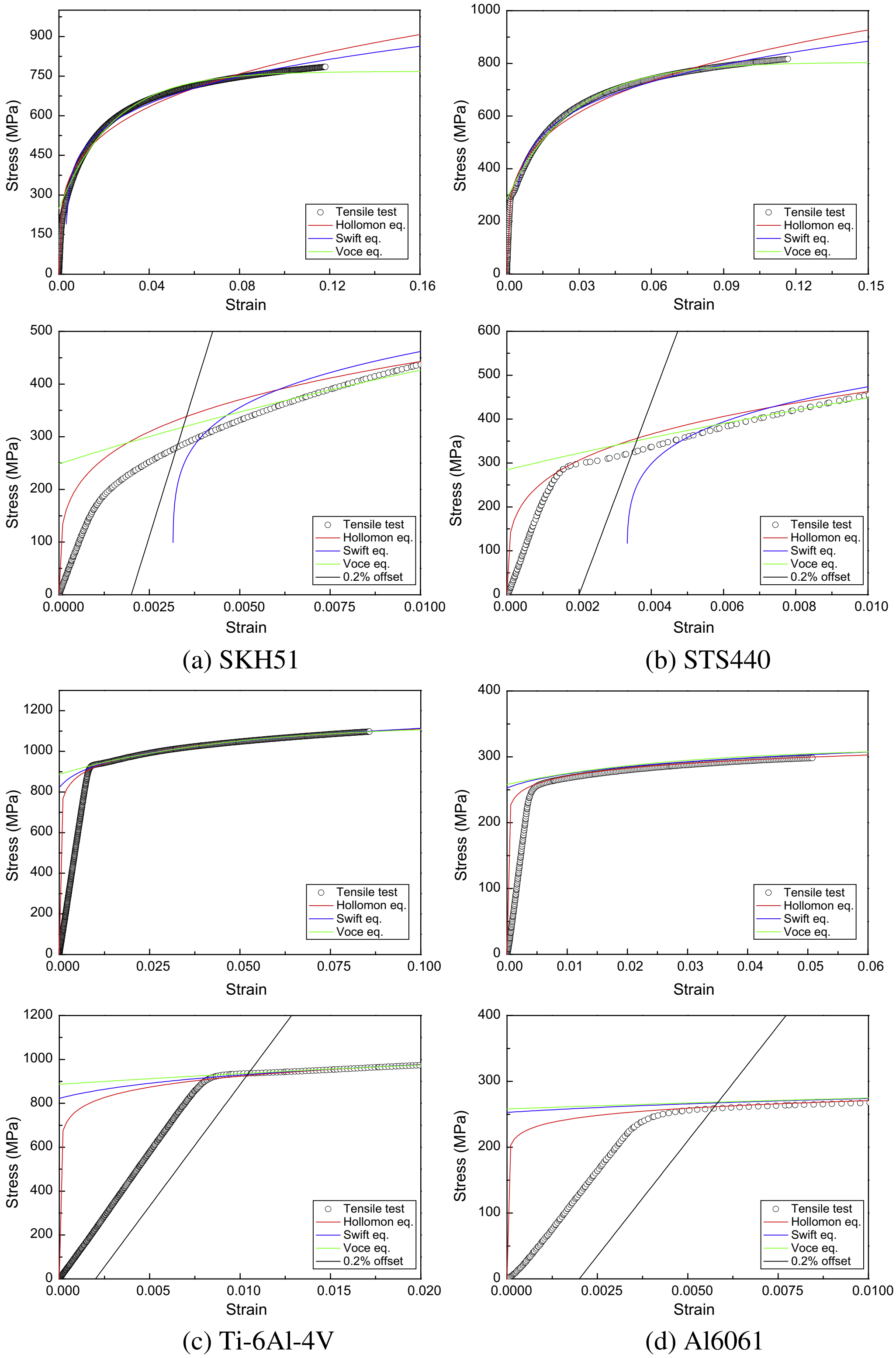

Fig. 1. True stress–strain curves and fitted curves by three constitutive equations for SKH51, STS440, Ti–6Al–4V, and Al6061.

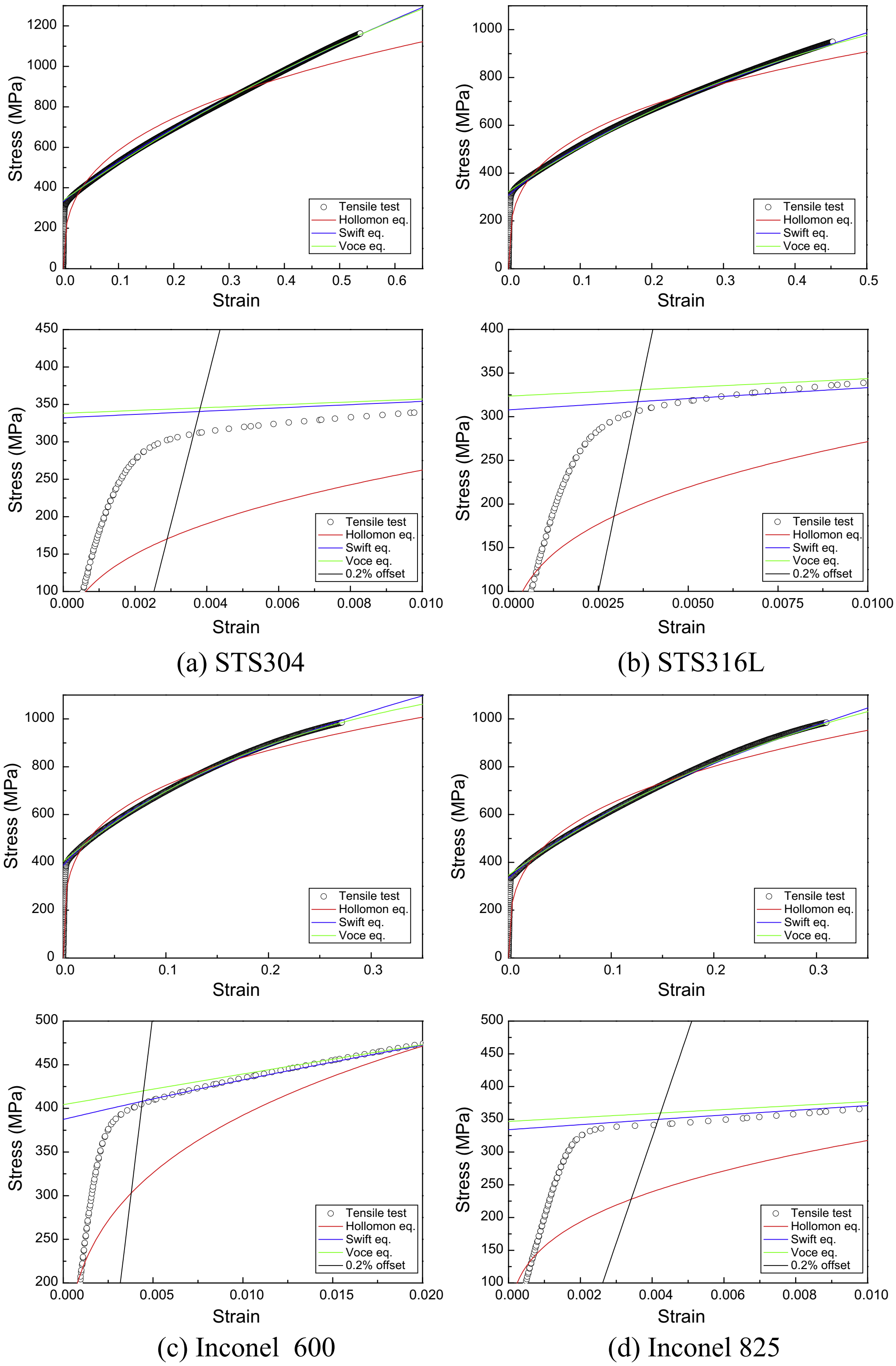

Fig. 2. True stress–strain curves and fitted curves by three constitutive equations for STS304, STS316L, Inconel 600, and Inconel 825.

used from yield strength to ultimate strength (plastic hardening region).

3. Results and discussion

Table 1 shows tensile properties of 27 metal alloys where stress and strain is listed in true stress and strain by converting engineering stress and strain. Elastic modulus was measured by ultrasonic method. Yield strength (σ_y) was measured by the intercept point of true stress–strain point with 0.2% offset line of elastic region. Ultimate tensile strength was determined as the true stress point which corresponds to maximum stress point in the engineering stress–strain curve. Table 2 shows parameters in the Hollomon, Swift, and Voce equations fitted to stress–strain points in plastic flow region, yield and ultimate tensile strengths analyzed by the constitutive equations, and their errors compared to the values in Table 1. Carbon steels, ferrite-based stainless steels, Ti alloys, and Al alloys show power-law-type strain hardening (Fig. 1), while austenite-based stainless steels and Ni alloys show linear-type strain hardening (Fig. 2).

Fig. 1 also presents stress–strain curves focusing low strain around yield points to investigate effects of constitutive equations on the determination of yield strength compared to the 0.2% offset method. The yield strengths determined by the 0.2% offset method together with the three constitutive equations show good agreements for Ti–6Al–4V and Al6061. As noted in Table 2, yield strengths analyzed by three constitutive equations for Ti alloys and Al alloys are in good agreement with those from the 0.2% offset method. Yield points cannot be determined when plastic flow curves are fitted by the Swift equation for SKH51, STS403, STS410, and STS440, as shown for SKH51 and STS440 in Fig. 1, because this equation exists only for strains higher than ε_0 in the Swift equation (Eq. (5)). One of the main purposes of the Swift equation is to describe plastic flow curve for materials with non-uniform elongation at yielding (Swift, 1952), so this equation may not be suitable for materials without non-uniform elongation at yielding. Evaluation of yield strengths using the Hollomon equation yields large underestimates for some stainless steels and Ni alloys. The Voce equation generally predicts the yield strengths with error about ±10%.

Fig. 2 shows true stress–strain points and three constitutive equations fitted to the points for austenitic stainless steels (STS304 and STS316L) and the Ni alloys (Inconel 600 and Inconel 825). They all show linear-type strain hardening (Fig. 2). The Hollomon equation does not describe plastic flow curves well for the materials with linear-type strain hardening, and tends to underestimate yield strengths (Swift, 1952; Ludwigson, 1971; Samuel and Rodriguez, 2005). On the other hand, the Swift and Voce equations show good correlations with true stress–strain points. The Swift equation is found to be the most suitable for evaluation of yield strengths for these metal alloys. Yield strengths analyzed by three equations are summarized in Table 2 for metal alloys with linear-type strain hardening, STS303, STS304L, STS310S, STS316, STS321, and STS347.

We investigate the accuracy of tensile ultimate strengths when the true stress–strain points are described by the three constitutive equations using instability conditions. Adopting Eq. (2) gives the instability conditions for the Swift and Voce equations (Swift, 1952; Voce, 1948, 1955) as

$$\varepsilon_u = n_s - \varepsilon_0 \tag{7}$$

for the Swift equation and

$$\varepsilon_u = -\frac{1}{\beta}\log\left(\frac{1}{(1+\beta)A}\right) \tag{8}$$

for the Voce equation. Table 2 also shows the difference in percent error between the ultimate tensile strengths analyzed by three constitutive equations and at maximum engineering stress directly measured from true stress–strain curves. The Hollomon equation generally yields the largest error in terms of ultimate tensile strength. The

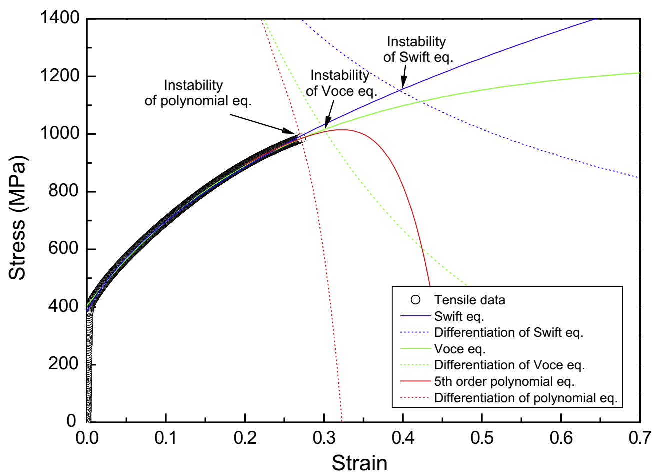

Fig. 3. Instability conditions of the Swift and Voce equations for Inconel 600.

Swift equation show good agreements for all expect SKH51 and Ni alloys. The Voce equation consistently shows good agreement for all kinds of metal alloys studied here.

To verify the concept of instability in tension, we use 5th-order polynomial fittings to true stress–strain points for Inconel 600 (Fig. 3). The instability condition is determined by the intercept of the stress–strain curve and its differential function. We found that the ultimate tensile strength point coincides with the instability condition in the polynomial fitting, implying that the instability condition is valid. Fig. 3 compares the Swift and Voce equations and their differential functions. The instability point derived by the Voce equation is closer to the actual one than that by the Swift equation. The instability condition is related not only to the stress–strain points but also their differential functions, so the constitutive equation should reflect the slope of the plastic flow curve correctly, especially in the uniform strain region. In the Hollomon and Swift equations, stress is expressed by a power-law function of strain, so the rate of strain does not become saturated. On the other hand, in the Voce equation, containing a decreasing exponential function of strain, the rate of stress increment decreases and saturates gradually as strain increases. For this reason, the Voce equation can be used to describe the decrement in the slope of the stress and strain curve near the necking point, unlike the Hollomon and Swift equations.

4. Conclusions

We examined the suitability of three constitutive equations, the Hollomon, the Swift, and the Voce equations, for 27 metal alloys. The Hollomon equation with two independent parameters did not describe plastic flow curves correctly for the materials with linear-type strain-hardening such as austenite-based stainless steels and Inconel alloy. Ultimate tensile strengths were not defined by instability conditions for some materials when true stress–strain curves are described by the Hollomon equation. The Swift equation with three independent parameters generally describes yield strengths well determined as the intercept of the linear elastic loading curve and plastic flow curve. However, the existence problem arises and the instability condition was violated for Inconel alloys. The Voce equation obeys the instability condition well and predicts yield strengths more accurately than the Hollomon and Swift equations.

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